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MISSING MIXING?

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ABSTRACT

Shear-generated turbulent mixing in stellar radiative regions (stably stratified) has recently been found to be too weak. This has stimulated the search for the missing mixing. We suggest that there may not be a missing mixing problem. Phenomenological models have underestimated its real strength, thus leading to a negative assessment of its real potential.

Subject headings: convection — hydrodynamics — stars: evolution — stars: interiors — stars: rotation

1. INTRODUCTION

It was suggested long ago that in the radiative-dominated regions of a star in which stratification is stable, turbulent mixing may be provided by mean shear (Zahn 1974). Several well-documented studies (Pinsonneault et al. 1989; Schatzman & Baglin 1991; Maeder 1997; Maeder & Meynet 1996; Pinsonneault 1997; Chaboyer, Demarque, & Pinsonneault 1995; Talon et al. 1997) have, however, concluded that such a mechanism provides too little mixing to explain stellar structure and evolution data. Some mixing is missing.

The suggestion of this Letter is that *there may not be a missing mixing problem*. Using state-of-the-art turbulence modeling, we show that shear-induced mixing is larger than previously thought and perhaps even sufficient to overcome the "missing mixing." Clearly, the last assertion can only be confirmed (or denied) by a specific stellar calculation that employs the results of this model. Before we present the quantitative arguments, we discuss the physics of the problem. There are two key processes. First, turbulence generated by a mean shear can only survive up to a critical Richardson number $\mathrm{Ri}_{\mathrm{cr}}$, above which stable stratification dominates and extinguishes turbulence. What is the true value of $\mathrm{Ri}_{\mathrm{cr}}$?

Second, radiative losses by the eddies weaken the stabilizing temperature gradient and thus help turbulence. We recall that $\mathrm{Ri} = N^2/\Sigma^2$, where N is the Brunt-Vaisala frequency; N^2 is proportional to the positive temperature gradient, while Σ is the mean shear that generates turbulence. Radiative losses erode the temperature gradient, lower N, and thus force the system toward the neutral case $\mathrm{Ri} \sim 0$, thus maximizing the shear-generated mixing. The two processes are clearly not independent, and one must also account for their interplay.

In spite of the apparent simplicity, proper accounting of the two effects is not simple. This can be easily understood in the case of radiative losses that damp turbulence potential energy and convective fluxes. The processes involve not the largest but the smallest scales, and that is where the difficulties lie. Large scales are long lived and, most importantly, have very low degrees of vorticity, thus making them easier to account for. In fact, bulk properties like convective fluxes that are dominated by the large scales were described acceptably well by simple, one-eddy theories like the mixing-length theory. Small scales are quite different in nature and considerably harder to describe, for in them reside the highest levels of vorticity. Contrary to the universal Kolmogorov spectrum describing medium-size eddies, the high wavenumber k (UV), the small-scale part of the energy spectrum is not universal.

Both large eddy simulation (LES) and one-point turbulent closure models are useless in this context. In the first case, present and foreseeable LESs barely (and often do not even) resolve the first scale of the Kolmogorov spectrum, let alone scales that are several orders of magnitude smaller and that are where most of the vorticity resides. In addition, the sub–grid scale models used to account for the unresolved scales are quite different from the traditional, Smagorinsky-like model used for shear-generated and/or unstable stratification. Similarly, the well-known one-point closure models that were successfully developed over the last 40 years do not provide the energy spectrum $E(k)$ but only its integral over all k's. Thus, the vorticity which is the integral of $k^2E(k)$ over all k's cannot be evaluated. The problem can also be presented in different terms. One needs to properly account for the Peclet number

$$\mathrm{Pe} = \frac{\nu_t}{\chi}, \tag{1}$$

which is the ratio of the turbulent viscosity to the radiative conductivity χ. The variable $\nu_t(k)$ for an eddy of size $\ell \sim k^{-1}$ is the key quantity; from the first model by Heisenberg (Batchelor 1970) to the most modern ones (Lesieur 1991), $\nu_t(k)$ is expressed as a UV property, since it is contributed by all eddies smaller than k^{-1}. Its general structure is thus of the form

$$\nu_t(k) \equiv \int_k^\infty \psi\,(k')\,dk', \tag{2}$$

and the challenge is to model the variable $\psi(k', \mathrm{Pe}, \mathrm{Ri})$, which is a function of Ri and Pe. We need to stress that the difficulty is not that of carrying out a calculation that includes Pe, which is mostly a bookkeeping problem (Townsend 1958a, 1958b); the real difficulty is to compute Pe, which in turn implies a knowledge of the UV part of the turbulent energy spectrum. This requires a turbulence model.

Fortunately, the inapplicability of both LES and one-point closures has been recently remedied by the renormalization group (RNG) techniques. As discussed elsewhere (Canuto & Dubovikov 1966), the RNG is an exact model for the UV portion of the eddy spectrum because it entails a finite number of irreducible Feynmann diagrams that can be summed exactly. This is precisely the region critical to the evaluation of the dissipation timescales in question. In practical terms, the RNG techniques provide the function $\psi(k, \mathrm{Pe}, \mathrm{Ri})$ of equation (2) and thus $\nu_t(k)$ and the required timescale $\tau^{-1} \sim \nu_t(k)k^2$ as a function of Pe and Ri.

Next, consider the question of $\mathrm{Ri}_{\mathrm{cr}}$. Thus far, all studies have assumed $\mathrm{Ri}_{\mathrm{cr}} = 1/4$. To avoid possible confusion, let us recall that such result does not follow from the work of Townsend (1958a, 1958b), which is often used in this context. The purpose of the latter was to show that *under* Pe < *1 conditions*, the effective Richardson number is RiPe rather than Ri itself, or that the effective Brunt-Vaisala frequency is not N but $N\mathrm{Pe}^{1/2} < N$. This weakens the role of stratification. Any complete turbulence model naturally reproduces this renormalization. The $\mathrm{Ri}_{\mathrm{cr}} = 1/4$ is strictly the result of linear stability analysis. There is, however, ample evidence from a variety of sources that turbulence exists quite past Ri = 1/4. Monin & Yaglom (1971) report up to $\mathrm{Ri}_{\mathrm{cr}} = 10$, as from work of G. I. Taylor. Modern data from laboratory and oceanography (Martin 1985; Smart 1988; Wang, Large, & McWilliams 1996) yield $\mathrm{Ri}_{\mathrm{cr}} = 1.4$–1.6, which is more than a factor of 5 larger than $\mathrm{Ri}_{\mathrm{cr}} = 1/4$. The problem thus reduces to that of finding a physical definition of $\mathrm{Ri}_{\mathrm{cr}}$. We suggest abandoning the linear stability approach which, irrespective of what it predicts, may have no bearing on a fully turbulent regime such as the one found in stars and adopts the following physical definition: $\mathrm{Ri}_{\mathrm{cr}}$ *is the value of* Ri *at which the turbulent kinetic energy vanishes.* From the pragmatic point of view, the definition only makes sense if one has a model of turbulence. The model that we employ has been tested on more than 80 turbulent statistics pertaining to a large variety of turbulent flows of very different natures, from convection to shear, two-dimensional, rotation, etc. In addition, when integrated over all k's, the model equations reproduce the one-point closure models that were constructed in the last 40 years.

Finally, and quite importantly, the model satisfies the following two requirements: when $\mathrm{Ri} \to \infty$ (no shear), it reproduces the standard formulae for convection; when $\mathrm{Ri} \to 0$, it reproduces well-known expressions for shear-induced turbulence.

2. NEW MODEL

The basic physical variables are $\overline{u_i u_j} = \tau_{ij}$ (Reynolds stresses), $K \equiv 1/2\tau_{ii}$ (turbulent kinetic energy), $\overline{u_i\theta} = h_i$ (convective fluxes), $\overline{\theta^2}$ (temperature variance), and the rates of dissipation of K and $\overline{\theta^2}$, ϵ and ϵ_θ. Here, u and θ are the fluctuating components of the velocity and temperature fields. The timescales discussed earlier are the ones that enter in the determination of ϵ_θ, since

$$\epsilon_\theta = \overline{\theta^2}\tau_\theta^{-1} \quad \text{and} \quad \tau_\theta = f(\mathrm{Pe}, \mathrm{Ri}). \tag{3}$$

The key function $f(\mathrm{Pe}, \mathrm{Ri})$ is provided by the RNG (Canuto & Dubovikov 1998, eq. [34]). The dynamic equations for the above variables have already been derived (Canuto 1994, eqs. [22]–[26]), and there is no reason to repeat them here. Rather, we discuss the main steps and the main results, leaving the details for a more extended publication.

First, neglecting the diffusion terms and taking the stationary limit, the above dynamic equations reduce to algebraic relations. If one takes the T and U fields as depending on z only,

$$\frac{\partial T}{\partial x_i} \to \frac{\partial T}{\partial z}\delta_{i3}, \quad \boldsymbol{U} = [U(z), V(z), 0], \tag{4}$$

the results simplify considerably. The Reynolds stresses and convective flux $F_c = c_p\rho\overline{w\theta}$ are given by the expressions

$$\overline{uw} = -\nu_T\frac{\partial U}{\partial z} \quad \text{and} \quad \overline{w\theta} = \chi_T\beta, \tag{5}$$

where $\beta = TH_p^{-1}(\nabla - \nabla_{\mathrm{ad}})$ is the superadiabatic gradient. The turbulent viscosity and conductivity are expressed as

$$(\nu_T, \chi_T) = 2\frac{K^2}{\epsilon}(S_\nu, S_h), \tag{6a}$$

where $S_{\nu,h}$ are dimensionless functions of

$$S_{\nu,h} = F(\tau N, \tau\Sigma, \mathrm{Pe}). \tag{6b}$$

Second, one assumes that $P = \epsilon$ (production = dissipation),

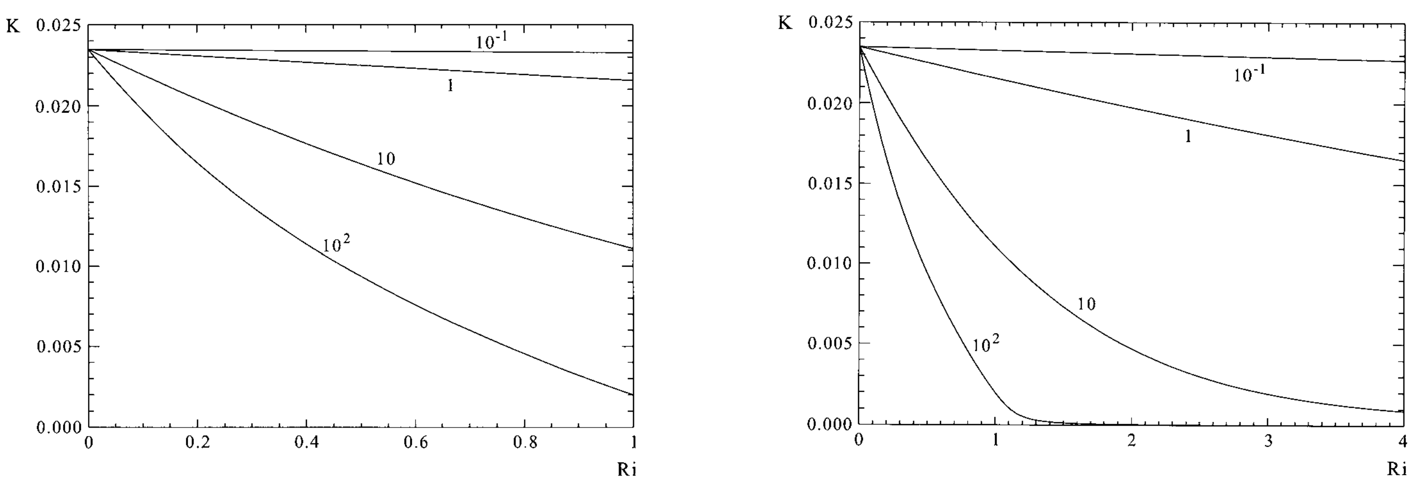

FIG. 1.—Turbulent kinetic energy K (in units of $4\ell_\epsilon^2\Sigma^2$) vs. Ri for different values of Pe_0 (defined in the text). The level of turbulence decreases as Ri increases. When radiative losses are important, Pe < *l*, stratification is weakened, and the slowdown is considerably reduced.

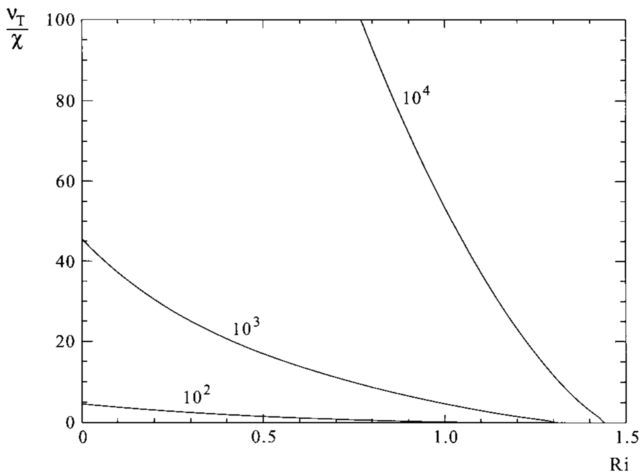

FIG. 2.—Momentum turbulent diffusivity ν_T (in units of χ) vs. Ri for different values of Pe_0. The standard model gives no momentum diffusivity past Ri = 1/4. In the present model, ν_T does not diverge at Ri $\rightarrow$ 0; its value depends on Pe_0 and is finite beyond Ri = 1/4.

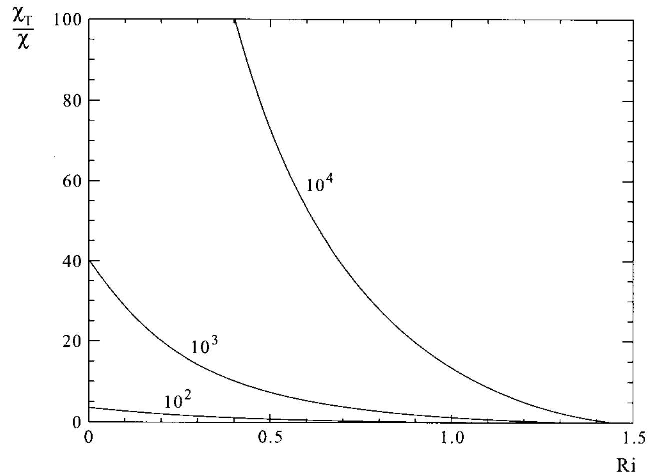

FIG. 3.—Same as in Fig. 2 but for the heat diffusivity χ_T. The standard model gives no heat diffusivity past Ri = 1/4.

where ($N^2 = -gT^{-1}\beta,\ \Sigma^2 = U^2_{,z} + V^2_{,z}$)

$$P = \nu_T\Sigma^2 - \chi_T N^2, \tag{6c}$$

while the local expression for ϵ is

$$\epsilon = \frac{K^{3/2}}{\ell_\epsilon}, \tag{6d}$$

where the mixing length ℓ_ϵ must be specified. Third, substituting equations (6a) and (6d) with P given by equation (6c), the $P = \epsilon$ relation gives the equation for $K(\mathrm{Ri, Pe})$:

$$K = 2\ell_\epsilon^2\Sigma^2(S_\nu - \mathrm{Ri}S_h). \tag{6e}$$

3. RESULTS

Using equation (6b), one can solve equation (6e). The result is shown in Figure 1. As one can see, even for large Pe_0 [$\mathrm{Pe}_0 = c_\epsilon^2\ell_\epsilon^2\Sigma\chi^{-1}$, $c_\epsilon = \pi(2/3\mathrm{Ko})^{3/2}$, where Ko is the Kolmogorov constant], no radiative losses, the present model yields turbulence quite past $\mathrm{Ri_{cr}} = 1/4$. The standard model has $K = 0$ beyond Ri = 1/4. In the new model, turbulence dies out only at $\mathrm{Ri_{cr}} = 1$–2, in agreement with oceanographic and laboratory data. As radiative losses by the eddies become important and Pe decreases, the value of $\mathrm{Ri_{cr}}$ increases and turbulence lives longer since the damping effect of the stabilizing temperature gradient is reduced. Thus, $\mathrm{Ri_{cr}}$ is not a universal value but depends on the radiative losses, that is, on Pe:

$$\mathrm{Ri_{cr}} = f(\mathrm{Pe}). \tag{7a}$$

This is indeed borne out by the present model. In Figures 2 and 3, we exhibit the turbulent viscosity ν_T and the turbulent conductivity χ_T versus Ri for different values of Pe_0. The standard model can be represented by

$$\frac{\nu_T}{\chi} = \frac{1}{12\mathrm{Ri}}, \quad \mathrm{Ri} < 1/4, \tag{7b}$$

$$\frac{\nu_T}{\chi} = 0, \quad \mathrm{Ri} > 1/4. \tag{7c}$$

Some comments are in order: first, as Ri $\rightarrow$ 0, equation (7b) yields a divergent result that is unphysical since Ri $\rightarrow$ 0 implies no stratification (neutral case), but it is known that ν_T is finite; second, equation (7c) gives zero mixing beyond Ri = 1/4; third, relations (7b) and (7c) are only valid for small Pe, but even so, they do not distinguish among different Pe's. By contrast, the present model is valid for arbitrary Pe, recovers the renormalization Ri $\rightarrow$ RiPe for small Pe's, has a Pe dependence, and exhibits a turbulent regime that lasts quite past the $\mathrm{Ri_{cr}} = 1/4$ limit.

4. CONCLUSIONS

Until this new model is tested in a specific stellar case, we cannot claim that the amount of "missing mixing" that is required by stellar data can all be provided by the new model. On the other hand, we have shown that the standard model substantially underestimates the mixing and that perhaps there is no need to search for other mechanisms. Shear-generated turbulence may be sufficient, if properly quantified.

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